ON SUMS OF INTEGRALS OF POWERS OF THE ZETA-FUNCTION IN SHORT INTERVALS

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Abstract. The modified Mellin transform $\mathcal{Z}_k(s) = \int_1^\infty |\zeta(\frac{1}{2}+ix)|^{2k}x^{-s}\,\mathrm{d}x$ ($k \in \mathbb{N}$) is used to obtain estimates for

$$\sum_{r=1}^{R}\int_{t_r-G}^{t_r+G}|\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t \quad (T < t_1 < \cdots < t_R < 2T),$$

where $t_{r+1}-t_r \ge G$ ($r = 1,\ldots,R-1$), $T^\varepsilon \le G \le T^{1-\varepsilon}$. These results can be used to derive bounds for the moments of $|\zeta(\frac{1}{2}+it)|$.

1. Introduction

The (modified) Mellin transforms

$$\mathcal{Z}_k(s) := \int_1^\infty |\zeta(\tfrac{1}{2}+ix)|^{2k}x^{-s}\,\mathrm{d}x \qquad (k\in\mathbb{N},\ \sigma = \Re\mathrm{e}\, s \ge c(k) > 1), \tag{1.1}$$

where $c(k)$ is such a constant for which the integral in (1.1) converges absolutely, play an important rôle in the theory of the Riemann zeta-function $\zeta(s)$ (see [1], [7], [9], [14] and [19] for some of the relevant works, which contain further references). The term "modified" Mellin transform seems appropriate, since customarily the Mellin transform of $f(x)$ is defined as

$$F(s) := \int_0^\infty f(x)x^{s-1}\,\mathrm{d}x \qquad (s = \sigma+it\in\mathbb{C}). \tag{1.2}$$

Note that the lower bound of integration in (1.1) is not zero, as it is in (1.2). The choice of unity as the lower bound of integration dispenses with convergence problems at that point, while the appearance of the factor x^{-s} instead of the

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customary x^{s-1} is technically more convenient. Also it may be compared with the discrete representation

$$\zeta^{2k}(s) \;=\; \sum_{n=1}^\infty d_{2k}(n)n^{-s} \qquad (\sigma > 1,\, k \in \mathbb{N}),$$

where $d_m(n)$ is the number of ways n may be written as a product of m factors; $d(n) \equiv d_2(n)$ is the number of divisors of n. Since we have (see [3, Chapter 8])

$$\int_0^T |\zeta(\tfrac12 + it)|^{2k}\,{\rm d}t \;\ll\; T^{(k+2)/4}\log^{C(k)}T \qquad (2 \le k \le 6; C(k) \ge 0),$$

it follows that the integral defining $\mathcal{Z}_k(s)$ is absolutely convergent for $\sigma > 1$ if $0 \le k \le 2$ and for $\sigma > (k+2)/4$ if $2 \le k \le 6$.

The function $\mathcal{Z}_k(s)$ is a special case of the multiple Dirichlet series

$$Z(s_1,\cdots,s_{2k},w) = \int_1^\infty \zeta(s_1+it)\cdots\zeta(s_k+it)\zeta(s_{k+1}-it)\cdots\zeta(s_{2k}-it)t^{-w}\,{\rm d}t \tag{1.3}$$

considered in a recent work of A. Diaconu, D. Goldfeld and J. Hoffstein [1]. Analytic properties of this function may be put to advantage to deal with the important problem of the analytic continuation of the function $\mathcal{Z}_k(s)$ itself. It is shown in [1] that (1.3) has meromorphic continuation (as a function of $2k+1$ complex variables) slightly beyond the region of absolute convergence, with a polar divisor at $w = 1$. It is also shown that (1.3) satisfies certain quasi-functional equations, which are used to obtain meromorphic continuation to an even larger region. Under the assumption that

$$Z(\tfrac12,\cdots,\tfrac12,w) \;\equiv\; \mathcal{Z}_k(w)$$

has holomorphic continuation to the region $\Re{\rm e}\, w \ge 1$ (except for the pole at $w = 1$ of order k^2+1), the authors derive the conjecture on the moments of the zeta-function on the critical line in the form

$$\int_0^T |\zeta(\tfrac12 + it)|^{2k}\,{\rm d}t \;=\; (c_k + o(1))T\log^{k^2}T \qquad (T \to \infty), \tag{1.4}$$

where $k \ge 2$ is a fixed integer and

$$c_k = \frac{a_k g_k}{\Gamma(1+k^2)},\; a_k = \prod_p (1-1/p)^{k^2}\sum_{j=0}^\infty d_k^2(p^j)p^{-j},\; g_k = (k^2)!\prod_{j=0}^{k-1}\frac{j!}{(j+k)!}. \tag{1.5}$$

The formulas (1.4)-(1.5) coincide with the well-known conjecture from Random Matrix Theory (see e.g., J. Keating and N. Snaith [17]) on the even moments of $|\zeta(\frac{1}{2}+it)|$.

In general one expects, for any fixed $k \in \mathbb{N}$,

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t = TP_{k^2}(\log T) + E_k(T) \tag{1.6}$$

to hold (see the author's monograph [4] for an extensive account), where it is generally assumed that

$$P_{k^2}(y) = \sum_{j=0}^{k^2} a_{j,k}y^j \tag{1.7}$$

is a polynomial in y of degree k^2 (the integral in (1.6) is $\gg_k T\log^{k^2} T$; see e.g., [3, Chapter 9]). The function $E_k(T)$ is to be considered as the error term in (1.7), namely one supposes that

$$E_k(T) \;=\; o(T) \qquad (T \to \infty). \tag{1.8}$$

So far (1.6)–(1.8) are known to hold only for $k = 1$ and $k = 2$ (see [3], [4] and [18]).

In case (1.6)–(1.8) hold, this may be used to obtain the analytic continuation of $\mathcal{Z}_k(s)$ to the region $\sigma \ge 1$ (at least). Indeed, by using (1.6)–(1.8) we have

$$\begin{aligned}\mathcal{Z}_k(s) &= \int_1^\infty |\zeta(\tfrac{1}{2}+ix)|^{2k}x^{-s}\,\mathrm{d}x = \int_1^\infty x^{-s}\,\mathrm{d}\left(xP_{k^2}(\log x) + E_k(x)\right)\\ &= \int_1^\infty (P_{k^2}(\log x) + P'_{k^2}(\log x))x^{-s}\,\mathrm{d}x - E_k(1) + s\int_1^\infty E_k(x)x^{-s-1}\,\mathrm{d}x.\end{aligned} \tag{1.9}$$

But for $\Re\mathrm{e}\, s > 1$ change of variable $\log x = t$ gives

$$\begin{aligned}&\int_1^\infty (P_{k^2}(\log x) + P'_{k^2}(\log x))x^{-s}\,\mathrm{d}x\\ &= \int_1^\infty \left\{\sum_{j=0}^{k^2} a_{j,k}\log^j x + \sum_{j=0}^{k^2-1}(j+1)a_{j+1,k}\log^j x\right\} x^{-s}\,\mathrm{d}x\\ &= \int_0^\infty \left\{\sum_{j=0}^{k^2} a_{j,k}t^j + \sum_{j=0}^{k^2-1}(j+1)a_{j+1,k}t^j\right\} \mathrm{e}^{-(s-1)t}\,\mathrm{d}t\\ &= \frac{a_{k^2,k}(k^2)!}{(s-1)^{k^2+1}} + \sum_{j=0}^{k^2-1}(a_{j,k}j! + a_{j+1,k}(j+1)!)(s-1)^{-j-1}.\end{aligned} \tag{1.10}$$

Hence inserting (1.10) in (1.9) and using (1.8) we obtain the analytic continuation of $\mathcal{Z}_k(s)$ to the region $\sigma \ge 1$. As we know (see [3], [4], [11] and [19]) that

$$\int_1^T E_1^2(t)\,\mathrm{d}t \ll T^{3/2}, \qquad \int_1^T E_2^2(t)\,\mathrm{d}t \ll T^2 \log^{22} T, \tag{1.11}$$

it follows on applying the Cauchy–Schwarz inequality to the last integral in (1.9) that (1.8)-(1.10) actually provides the analytic continuation of $\mathcal{Z}_1(s)$ to the region $\Re\mathrm{e}\, s > 1/4$, and of $\mathcal{Z}_2(s)$ to $\Re\mathrm{e}\, s > 1/2$, but is is actually known that $\mathcal{Z}_1(s)$ (resp. $\mathcal{Z}_2(s)$) has meromorphic continuation to $\mathbb{C}$. For this, see M. Jutila [16] when $k = 1$ and Y. Motohashi [19] when $k = 2$.

The preceding discussion shows one of the several aspects of the connection between the function $\mathcal{Z}_k(s)$ and power moments of $|\zeta(\frac{1}{2} + it)|$. The aim of this paper is to bring forth some results concerning the mean values of $\mathcal{Z}_k(s)$ and sums of integrals of the form

$$\sum_{r=1}^{R} \int_{t_r-G}^{t_r+G} |\zeta(\tfrac{1}{2} + it)|^{2k}\,\mathrm{d}t \quad (T < t_1 < \cdots < t_R < 2T) \tag{1.12}$$

for well-spaced points t_r which satisfy $t_{r+1} - t_r \ge G$ $(r = 1, \ldots, R - 1)$, where $G = G(T)$ is parameter satisfying $T^\varepsilon \le G \le T^{1-\varepsilon}$, while here and later ε denotes arbitrarily small constants, not necessarily the same ones at each occurrence.

Bounds for sums of the type (1.12) with $k = 2$ were obtained first by H. Iwaniec [15], who showed that the left-hand side of (1.12) in this case is bounded by $T^\varepsilon(RG + R^{1/2}TG^{-1/2})$ for $T^{1/2} \le G \le T$. Later the author and Y. Motohashi [12] replaced T^ε by a log-power. In their work [11] the range for G was relaxed to $\log T \ll G \ll T/\log T$, and the result was generalized. Further generalizations and results were obtained by the author in [5].

One of the applications involving sums of the form (1.12) consists of obtaining upper bounds for moments of $|\zeta(\frac{1}{2} + it)|$. Namely one counts (see e.g., Chapter 8 of [3]) S, the number of well-spaced points τ_s in $[T, 2T]$ $(\tau_{s+1} - \tau_s \ge 1)$ such that $|\zeta(\frac{1}{2} + i\tau_s)| \ge V$ $(\ge T^\varepsilon)$. Then, by Theorem 1.2 of [4], it follows that for any fixed $k \in \mathbb{N}$ we have

$$V^{2k} \le |\zeta(\tfrac{1}{2}+i\tau_s)|^{2k} \ll \log T \int_{\tau_s-1/3}^{\tau_s-1/3} |\zeta(\tfrac{1}{2}+iu)|^{2k}\,\mathrm{d}u \qquad (s = 1, 2, \ldots, S), \tag{1.13}$$

and one groups integrals on the right hand side of (1.13) into sums of R integrals over intervals $[t_r - G,\, t_r + G]$ with $t_{r+1} - t_r \ge G$ (by considering separately r with even and odd indices). In this way sums of the type (1.12) arise, and their estimation leads to estimates for $\int_0^T |\zeta(\frac{1}{2} + it)|^{2k}\,\mathrm{d}t$, which is one of the central problems in the theory of $\zeta(s)$.

2. Statement of results

We begin with

THEOREM 1. *Let* $T < t_1 < t_2 < \ldots < t_R < 2T$, $t_{r+1} - t_r \ge G$ *for* $r = 1, \ldots, R-1$. *If, for fixed* $m, k \in \mathbb{N}$, *we have*

$$\int_T^{2T} \left(\frac{1}{G}\int_{t-G}^{t+G} |\zeta(\tfrac12 + iu|^{2k}\,\mathrm{d}u\right)^m \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon} \tag{2.1}$$

for $G = G(T) \ge T^{\alpha_{k,m}}$ *and* $0 \le \alpha_{k,m} \le 1$, *then*

$$\sum_{r=1}^R \int_{t_r-G}^{t_r+G} |\zeta(\tfrac12 + it)|^{2k}\,\mathrm{d}t \ll_\varepsilon (RG)^{\frac{m-1}{m}} T^{\frac{1}{m}+\varepsilon}. \tag{2.2}$$

The second result, although it could be easily generalized to sums of the form (1.12), deals with sums of fourth powers. This is because we have satisfactory results on the mean square of $\mathcal{Z}_k(s)$ so far only for $k = 1, 2$. The result is

THEOREM 2. *Let* $T < t_1 < t_2 < \ldots < t_R < 2T$, $t_{r+1} - t_r \ge G$ *for* $r = 1, \ldots, R-1$. *Then, for fixed* $\frac12 < \sigma < 1$, *we have*

$$\sum_{r=1}^R \int\limits_{t_r-G}^{t_r+G} |\zeta(\tfrac12 + it)|^4\,\mathrm{d}t \ll_\varepsilon RG\log^4 T + \Bigl(RGT^{2\sigma-1} \int\limits_0^{T^{1+\varepsilon}G^{-1}} |\mathcal{Z}_2(\sigma + it)|^2\,\mathrm{d}t\Bigr)^{1/2}. \tag{2.3}$$

The estimate (2.3) clearly shows the importance of the estimation of $\mathcal{Z}_2(s)$. Concerning the pointwise estimation of $\mathcal{Z}_2(s)$, we have (see the author's work [9])

$$\mathcal{Z}_2(\sigma + it) \ll_\varepsilon t^{\frac43(1-\sigma)+\varepsilon} \qquad (\tfrac12 < \sigma \le 1;\ t \ge t_0 > 0), \tag{2.4}$$

and it was conjectured in [7] that the exponent on the right-hand side of (2.4) can be replaced by $1/2 - \sigma$. This conjecture is very strong, as it was shown in [7] that it implies

$$\int_0^T |\zeta(\tfrac12 + it)|^8\,\mathrm{d}t \ll_\varepsilon T^{1+\varepsilon}, \quad E_2(T) \ll_\varepsilon T^{1/2+\varepsilon}, \tag{2.5}$$

where $E_2(T)$ (cf. (1.6)) is the error term (see [4], [6], [19]) in the asymptotic formula for the fourth moment of $|\zeta(\frac12 + it)|$. Both estimates in (2.5) are, up to "ε", known to be best possible.

For the mean square bounds of $\mathcal{Z}_2(s)$ we have the following. It was proved in by M. Jutila. Y. Motohashi and the author in [14] that

$$\int_1^T |\mathcal{Z}_2(\sigma+it)|^2\,\mathrm{d}t \ll_\varepsilon T^\varepsilon\left(T+T^{\frac{2-2\sigma}{1-c}}\right) \qquad (\tfrac12<\sigma\le 1), \tag{2.6}$$

and we also have unconditionally

$$\int_1^T |\mathcal{Z}_2(\sigma+it)|^2\,\mathrm{d}t \ll T^{\frac{10-8\sigma}{3}}\log^C T \qquad (\tfrac12<\sigma\le 1,\, C>0). \tag{2.7}$$

The constant c appearing in (2.6) is defined by $E_2(T)\ll_\varepsilon T^{c+\varepsilon}$, and it is known (see e.g., [4] or [12]) that $\frac12\le c\le\frac23$. In (2.6)–(2.7) σ is assumed to be fixed, as $s=\sigma+it$ has to stay away from the $\frac12$-line where $\mathcal{Z}_2(s)$ has poles. Lastly, the author [10] proved that, for $\frac56\le\sigma\le\frac54$ we have,

$$\int_1^T |\mathcal{Z}_2(\sigma+it)|^2\,\mathrm{d}t \ll_\varepsilon T^{\frac{15-12\sigma}{5}+\varepsilon}. \tag{2.8}$$

The lower limit of integration in (2.6)–(2.8) is unity, because of the pole $s=1$ of $\mathcal{Z}_2(s)$. By taking $c=2/3$ in (2.6) and using the convexity of mean values (see e.g., [3, Lemma 8.3]) it follows that

$$\int_1^T |\mathcal{Z}_2(\sigma+it)|^2\,\mathrm{d}t \ll_\varepsilon T^{\frac{7-6\sigma}{2}+\varepsilon} \qquad (\tfrac12<\sigma\le\tfrac56). \tag{2.9}$$

Note that (2.8) and (2.9) combined provide the sharpest known bounds for the mean square of $\mathcal{Z}_2(s)$ in the whole range $\frac12<\sigma\le\frac56$.

Corollary 1. *We have*

$$\int_0^T |\zeta(\tfrac12+it)|^{12}\,\mathrm{d}t \ll_\varepsilon T^{2+\varepsilon}. \tag{2.10}$$

This follows from (2.3) and (2.7) with $\sigma=1/2+\varepsilon$, giving Iwaniec's bound

$$\sum_{r=1}^R \int_{t_r-G}^{t_r+G} |\zeta(\tfrac12+it)|^4\,\mathrm{d}t \ll_\varepsilon T^\varepsilon(RG+R^{1/2}TG^{-1/2}),$$

and then taking $k=2, G=VT^{-\varepsilon}$ in conjunction with (1.13). One immediately obtains $R\ll_\varepsilon T^{2+\varepsilon}V^{-12}$, and (2.10) follows. This result (with $\log^{17}T$ replacing

T^ε) is due to D.R. Heath-Brown [2], and still represents the strongest bound concerning high moments of $|\zeta(\frac12+it)|$.

In obtaining the analytic continuation and bounds for $\mathcal{Z}_2(s)$ in [14], the authors considered the function

(2.11)
$$Z_\xi(s) := \int\limits_1^\infty J_2(x;x^\xi)x^{-s}\,\mathrm{d}x,\ J_k(x;G) := \frac{1}{\sqrt{\pi}G}\int\limits_{-\infty}^\infty |\zeta(\tfrac12+ix+iu)|^{2k}\mathrm{e}^{-(u/G)^2}\,\mathrm{d}u,$$

where $k\in\mathbb{N}$, $0<\xi\le 1$, and initially $\Re\mathrm{e}\, s>1$. Because of the smooth Gaussian weight in (2.11) the function $Z_\xi(s)$ is in many aspects less difficult to deal with than the function $\mathcal{Z}_2(s)$ itself, especially in view of the spectral expansion of $J_2(x;G)$ obtained by Y. Motohashi (see [18] and [19]). Moreover, by Mellin inversion and Parseval's formula for Mellin transforms, one can connect bounds for $Z_\xi(s)$ to the left-hand side of (2.1) when $k=m=2$, and hence indirectly to power moments of $|\zeta(\frac12+it)|$. Therefore it seems of interest to obtain bounds for $Z_\xi(s)$, especially if they improve on the existing bounds for $\mathcal{Z}_2(s)$. In this direction we shall prove in this work a result which is stronger than the analogous bound (2.4) for $\mathcal{Z}_2(s)$. This is

THEOREM 3. *If σ and ξ are fixed, then*

$$Z_\xi(\sigma+it) \ll_\varepsilon |t|^{1-\sigma+\varepsilon} \qquad (\tfrac12<\sigma\le 1,\ \tfrac13\le\xi\le 1). \tag{2.12}$$

3. Proof of Theorem 1 and Theorem 2

We begin with the proof of Theorem 1. Set $L_k(t,G)=\int_{t-G}^{t+G}|\zeta(\frac12+iu)|^{2k}\,\mathrm{d}u$. Note that if $\mu(\cdot)$ denotes measure, the bound

$$\mu\Big(t\in[T,\,2T]\,:\,L_k(t,G)\ge GU\Big) \ll_\varepsilon T^{1+\varepsilon}U^{-m} \qquad (U>0) \tag{3.1}$$

follows from the assumption (2.1). We fix $G=G(T)$ and divide the sum over r in (2.2) into $O(\log T)$ subsums where $GU<L_k(t_r,G)\le 2UG$. Then, for U_0 ($\gg 1$) to be determined later, we have

$$\begin{aligned}
\sum_{r=1}^R L_k(t_r,G) &\ll GRU_0 + \log T\max_{U\ge U_0}\sum_{r,GU<L_k(t_r,G)\le 2GU} L_k(t_r,G)\\
&\ll GRU_0 + GU\log T\max_{U\ge U_0}\sum_{r,GU<L_k(t_r,G)\le 2GU} 1\\
&\ll_\varepsilon GRU_0 + \log T\max_{U\ge U_0} T^{1+\varepsilon}U^{1-m}\\
&\ll_\varepsilon GRU_0 + T^{1+\varepsilon}U_0^{1-m}.
\end{aligned}$$

Here we used the condition that $m \ge 1$ and the bound

$$\sum_{r, L_k(t_r,G) > GU} 1 \ll_\varepsilon T^{1+\varepsilon} U^{-m} G^{-1}. \tag{3.2}$$

To see this, note that if $L_k(t_r, G) > GU$, then

$$L_k(t, 2G) \ge L_k(t_r, G) > GU \qquad (\text{for } |t - t_r| \le G).$$

As we can split the sequence of points $\{t_r\}$ into five subsequences, say $\{t'_r\}$, such that $|t'_{r_1} - t'_{r_2}| \ge 5G$ for $r_1 \ne r_2$, we see that

$$G \sum_{r, L_k(t_r,G) > GU} 1 \ll \mu\Bigl(t \in [T,\, 2T] \,:\, L_k(t, 2G) \ge GU\Bigr),$$

and (3.2) follows from (3.1). The choice

$$U_0 \;=\; \left(\frac{T}{RG}\right)^{1/m} \qquad (\gg 1\,)$$

yields

$$\sum_{r=1}^{R} L_k(t_r, G) \;\ll_\varepsilon\; T^{1/m+\varepsilon} R^{1-1/m} G^{1-1/m},$$

which is our assertion (2.2).

Corollary 2. *If the hypotheses of Theorem 1 hold, then we have*

$$\int_0^T |\zeta(\tfrac12 + it)|^{2km}\,\mathrm{d}t \;\ll_\varepsilon\; T^{1+(m-1)\alpha_{k,m}+\varepsilon}. \tag{3.3}$$

This follows from (1.13), analogously to Corollary 1. Observe that ($G = x^\xi, Q_4 = P_4 + P_4'$; see (1.6)))

$$\begin{aligned} J_2(x;G) &= \frac{1}{\sqrt{\pi}G}\int_{-\infty}^{\infty}\Bigl\{Q_4(\log(x+u) + \frac{\mathrm{d}}{\mathrm{d}u}E_2(x+u))\mathrm{e}^{-(u/G)^2}\,\mathrm{d}u\Bigr\} \\ &= O(\log^4 x) + \frac{2}{\sqrt{\pi}G^3}\int_{-\infty}^{\infty} uE_2(x+u)\mathrm{e}^{-(u/G)^2}\,\mathrm{d}u. \end{aligned}$$

Hence using the second bound in (1.11) it follows that (2.1), for $k = m = 2$, holds with $\alpha_{2,2} = \frac12$. By (3.3) this leads to the bound

$$\int_0^T |\zeta(\tfrac12 + it)|^8\,\mathrm{d}t \;\ll_\varepsilon\; T^{3/2+\varepsilon},$$

which is (up to "ε", see Chapter 8 of [3]) the sharpest one known.

We pass now to the proof of Theorem 2. By the inversion formula for the modified Mellin transform (see Lemma 1 of the author's paper [7]) we have

$$|\zeta(\tfrac{1}{2}+ix)|^4 = \frac{1}{2\pi i}\int_{(1+\varepsilon)} \mathcal{Z}_2(s)x^{s-1}\,\mathrm{d}s \qquad (x>1). \tag{3.4}$$

In (3.4) we replace the line of integration by the contour $\mathcal{L}$, consisting of the same straight line from which the segment $[1+\varepsilon-i,\,1+\varepsilon+i]$ is removed and replaced by a circular arc of unit radius, lying to the left of the line, which passes over the pole $s=1$ of the integrand. By the residue theorem we deduce from (3.1) that

$$|\zeta(\tfrac{1}{2}+ix)|^4 = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{Z}_2(s)x^{s-1}\,\mathrm{d}s + Q_4(\log x) \qquad (x>1)$$

holds, where we have set (cf. (1.6) with $k=2$)

$$Q_4(\log x) = P_4(\log x) + P_4'(\log x).$$

Therefore, for a suitable constant c satisfying $\frac{1}{2}<c<1$, we have

$$|\zeta(\tfrac{1}{2}+ix)|^4 = \frac{1}{2\pi i}\int_{(c)} \mathcal{Z}_2(s)x^{s-1}\,\mathrm{d}s + Q_4(\log x) \qquad (x>1), \tag{3.5}$$

where $\int_{(c)}$ denotes integration over the line $\Re\mathrm{e}\, s = c$. Let now $\varphi_r(x)$ (≥ 0) be a smooth function supported in $[t_r-2G,\, t_r+2G]$ such that $\varphi_r(x)=1$ when $x\in[t_r-G,\, t_r+G]$, so that

$$\varphi_r^{(m)}(x) \ll_{r,m} G^{-m} \qquad (r=1,\dots R,\ m=0,1,2,\dots). \tag{3.6}$$

Analogously as in the proof of Theorem 1, we can split the sequence $\{t_r\}$ into five subsequences $\{t_r'\}$ such that that $|t_{r_1}'-t_{r_2}'|\ge 5G$ for $r_1\ne r_2$. If we multiply (3.5) by $\varphi_r(x)$, integrate and sum, we see (writing again t_r for t_r') that the left hand side of (2.3) is majorized by five sums of the type

$$\begin{aligned}&\sum_{r\le R}\int_{t_r-2G}^{t_r+2G}\varphi_r(x)|\zeta(\tfrac{1}{2}+ix)|^4\,\mathrm{d}x = O(RG\log^4 T)+\\ &+\sum_{r\le R}\frac{1}{2\pi i}\int_{(c)}\mathcal{Z}_2(s)\Big(\int_{t_r-2G}^{t_r+2G}\varphi_r(x)x^{s-1}\,\mathrm{d}x\Big)\,\mathrm{d}s,\end{aligned} \tag{3.7}$$

the integrals on the left-hand side of (3.7) being taken over disjoint intervals. Integrating by parts the integral over x in (3.7) m times, it follows that it equals

$$(3.8)\qquad (-1)^m \int_{t_r-2G}^{t_r+2G} \varphi_r^{(m)}(x)\frac{x^{s+m-1}}{s(s+1)\ldots(s+m-1)}\,\mathrm{d}x \ll_{r,m} \frac{T^{\sigma+m-1}}{G^m(1+|t|)^m}.$$

We can write

$$\sum_{r\le R}\int_{t_r-2G}^{t_r+2G} \varphi_r(x)x^{s-1}\,\mathrm{d}x = \int_{T/2}^{5T/2}\Phi(x)x^{s-1}\,\mathrm{d}x,$$

where $\Phi(x)$ equals $\varphi_r(x)$ in $[t_r-2G,\, t_r+2G]$, and otherwise it is equal to zero. The bound in (3.8) shows that the portion of the integral in (3.7) over s for which $|t|\ge T^{1+\varepsilon}G^{-1}$ is negligibly small (i.e., $\ll T^{-A}$ for any given constant $A>0$), provided that $m=m(\varepsilon,A)$ is a sufficiently large integer.

Thus the left-hand side of (2.3) is, for fixed $\frac12<\sigma<1$,

$$\begin{aligned}
&\ll RG\log^4 T + \int_{-T^{1+\varepsilon}G^{-1}}^{T^{1+\varepsilon}G^{-1}} |\mathcal{Z}_2(\sigma+it)|\left|\int_{T/2}^{5T/2}\Phi(x)x^{s-1}\,\mathrm{d}x\right|\mathrm{d}t\\
&\ll RG\log^4 T + \left(\int_0^{T^{1+\varepsilon}G^{-1}} |\mathcal{Z}_2(\sigma+it)|^2\,\mathrm{d}t\right)^{1/2}\left(\int_{T/2}^{5T/2}\Phi^2(x)x^{2\sigma-1}\,\mathrm{d}x\right)^{1/2}\\
&\ll RG\log^4 T + \left(\int_0^{T^{1+\varepsilon}G^{-1}} |\mathcal{Z}_2(\sigma+it)|^2\,\mathrm{d}t\right)^{1/2}(RGT^{2\sigma-1})^{1/2},
\end{aligned}$$

which is the assertion of Theorem 2. Here we used, beside the Cauchy-Schwarz inequality, the estimation

$$\int_{T/2}^{5T/2}\Phi^2(x)x^{2\sigma-1}\,\mathrm{d}x \le \int_{T/2}^{5T/2}\Phi(x)x^{2\sigma-1}\,\mathrm{d}x \ll RGT^{2\sigma-1},$$

and the following (this is Lemma 4 of [7])

LEMMA 1. *Suppose that $g(x)$ is a real-valued, integrable function on $[a,b]$, a subinterval of $[2,\infty)$, which is not necessarily finite. Then*

$$\int_0^T\left|\int_a^b g(x)x^{-s}\,\mathrm{d}x\right|^2\mathrm{d}t \le 2\pi\int_a^b g^2(x)x^{1-2\sigma}\,\mathrm{d}x \quad (s=\sigma+it\,, T>0,\, a<b).$$

This completes the proof of Theorem 2.

4. The proof of Theorem 3

The estimation of $Z_\xi(s)$ was indirectly carried out in [7] and [9] by the author, in the process of the estimation of the function $\mathcal{Z}_2(s)$. This function bears resemblance to the function $\mathcal{Z}_2(s)$, and it also has a pole of order five at $s=1$, and infinitely many poles on the line $\Re e\, s = \frac{1}{2}$. For $\mathcal{Z}_2(s)$ Y. Motohashi (see [19]) showed that it has meromorphic continuation over $\mathbb{C}$. In the half-plane $\sigma = \Re e\, s > 0$ it has the following singularities: the pole $s=1$ of order five, simple poles at $s = \frac{1}{2} \pm i\kappa_j$ ($\kappa_j = \sqrt{\lambda_j - \frac{1}{4}}$) and poles at $s = \frac{1}{2}\rho$, where ρ denotes complex zeros of $\zeta(s)$. Here as usual $\{\lambda_j = \kappa_j^2 + \frac{1}{4}\} \cup \{0\}$ is the discrete spectrum of the non-Euclidean Laplacian acting on $SL(2,\mathbb{Z})$-automorphic forms (see [19, Chapters 1–3] for a comprehensive account of spectral theory and the Hecke L-functions).

The estimation of $Z_\xi(s)$ reduces to the estimation of $O(\log t)$ finite integrals of the form

$$\int_{X/2}^{5X/2} \sigma(x) J_2(x; x^\xi) x^{-s}\, \mathrm{d}x, \tag{4.1}$$

where ($t > t_0 > 0$ is assumed) as in Section 3 of [7] $t^{1-\varepsilon} \ll X \ll t^A$ ($A = A(\sigma) > 0$) holds, and $\sigma(x)$ (≥ 0) is a smooth function supported in $[X/2,\, 5X/2]$, which equals unity in $[X, 2X]$. For $J_2(x; x^\xi)$ we use Y. Motohashi's spectral decomposition (see [19]), which we state here as

LEMMA 2. *If $J_2(x;x^\xi)$ is defined by* (2.11), *then we have*

$$J_2(T;T^\xi) = I_{2,r}(T,T^\xi) + I_{2,h}(T,T^\xi) + I_{2,c}(T,T^\xi) + I_{2,d}(T,T^\xi). \tag{4.2}$$

Here $I_{2,r}$ is an explicit main term, the contribution of $I_{2,h}$ is small,

$$I_{2,c}(T,T^\xi) = \pi^{-1} \int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2}+ir)|^6}{|\zeta(1+2ir)|^2} \Lambda(r;T,T^\xi)\, \mathrm{d}r,$$

$$I_{2,d}(T,T^\xi) = \sum_{j=1}^{\infty} \alpha_j H_j^3(\tfrac{1}{2}) \Lambda(\kappa_j;T,T^\xi), \tag{4.3}$$

where

$$\begin{aligned} \Lambda(r;T,T^\xi) = \tfrac{1}{2}\mathrm{Re}\Bigg\{ &\left(1 + \frac{i}{\sinh \pi r}\right) \Xi(ir;T,T^\xi) \\ &+ \left(1 - \frac{i}{\sinh \pi r}\right) \Xi(-ir;T,T^\xi)\Bigg\} \quad (r \in \mathbb{R}) \end{aligned} \tag{4.4}$$

with

$$\Xi(ir;T,T^\xi) = \frac{\Gamma^2(\frac12+ir)}{\Gamma(1+2ir)}\int_0^\infty (1+y)^{-\frac12+iT}y^{-\frac12+ir} \tag{4.5}$$
$$\times \exp\left(-\tfrac14 T^{2\xi}\log^2(1+y)\right)F(\tfrac12+ir,\tfrac12+ir;1+2ir;-y)\,\mathrm{d}y,$$

and F is the hypergeometric function.

The contribution of the main term $I_{2,r}$ in (4.2) (of order $\ll \log^4 T$) to (4.1) is small if one uses integration by parts and $\sigma^{(m)}(x) \ll_m X^{-m}$ $(m \ge 0)$. The same is true of the contribution of the continuous spectrum $I_{2,c}$, if one uses the bounds for $\Xi(ir;T,T^\xi)$ in Chapter 5 of [19]. The main contribution comes from $I_{2,d}$ (the "discrete spectrum") in (4.2), and the problem reduces to the asymptotic evaluation of the functions $\Lambda(r;T,T^\xi)$ and $\Xi(ir;T,T^\xi)$ in (4.4) and (4.5), respectively. This task was carried out in detail in the recent work of A. Ivić–Y. Motohashi [13]. In particular, we invoke the discussion in Section 5 of this paper. The major contribution to (4.1) of $\Xi(ir;T,T^\xi)$, by equation (5.14) of [13] turns out to be a multiple of

$$\int_{X/2}^{5X/2}\sigma(x)x^{-s}\Big(\sum_{\kappa_j\le X^{1-\xi}\log X}\alpha_j H_j^3(\tfrac12)\,I_\xi(x,\kappa_j)\Big)\,\mathrm{d}x, \tag{4.6}$$

where, for any fixed integer N, $G = x^\xi$ and effectively computable constants c_j,
(4.7)

$$I_\xi(x,\kappa_j) := x^{-1/2}\kappa_j^{-1/2}\exp\left\{-\tfrac14 G^2\log^2(1+y_0) + i\mathcal{F}(y_0) - i\kappa_j\log 4\right\}$$
$$= x^{-1/2}\kappa_j^{-1/2}\times$$
$$\exp\left\{-\tfrac14 G^2\log^2(1+y_0) + i\kappa_j\log\left(\frac{\kappa_j}{4ex}\right) + i\sum_{j=3}^{N} c_j\kappa_j^j x^{1-j} + O_N(\kappa_j^{N+1}x^{-N})\right\}.$$

This is understood in the following sense: the remaining terms in the evaluation of the relevant expression are either negligible (smaller than X^{-A} for any constant $A > 0$), or similar in nature to (4.7) (meaning that the oscillating exponential factor is the same, which is crucial), only of the lower order of magnitude than the corresponding terms in (4.7). The remaining notation is as follows. We have

$$y_0 = \frac{\kappa_j}{x}\left(\sqrt{1+\frac{\kappa_j^2}{4x^2}} + \frac{\kappa_j}{2x}\right), \tag{4.8}$$

so that $y_0 \sim \kappa_j/x$ as $x \to \infty$ in the relevant range. Moreover, we have

$$\mathcal{F}(y_0) = \kappa_j\log y_0 - 2\kappa_j\log\left(\frac{1+\sqrt{1+y_0}}{2}\right) - x\log(1+y_0). \tag{4.9}$$

The term $\exp\Big(O_N(\kappa_j^{N+1}x^{-N})\Big)$ in (4.7) is expanded into a power series. If we take N sufficiently large, then only the first term unity will make a non-negligible contribution. Hence instead of (4.6) we need to estimate

$$\sum_{\kappa_j\le X^{1-\xi}\log X}\alpha_j\kappa_j^{-1/2}H_j^3(\tfrac12)\int_{X/2}^{5X/2}x^{-1/2-i\kappa_j-s}\Big(\sigma(x)L_\xi(x,\kappa_j)\Big)\,\mathrm{d}x, \tag{4.10}$$

say, where

$$L_\xi(x,\kappa_j):=\exp\left\{-\tfrac14 G^2\log^2(1+y_0)+i\kappa_j\log\left(\frac{\kappa_j}{4\mathrm{e}}\right)+i\sum_{\ell=3}^{N}c_\ell\kappa_j^\ell x^{1-\ell}\right\}. \tag{4.11}$$

We integrate (4.10) many times by parts, using (4.11) and the facts that $\sigma(X/2)=\sigma(5X/2)=0$ and $\sigma^{(m)}(x)\ll_m X^{-m}$ for $m\ge 0$. Thus, since the integral of $x^{-1/2-i\kappa_j-s}$ is

$$\frac{x^{1/2-i\kappa_j-s}}{\frac12-i\kappa_j-s}\qquad(\sigma>\tfrac12,\ \tfrac12-i\kappa_j-s\ne 0)$$

and

$$(\sigma(x)I_1)'\ll X^{-1}+\kappa_j^3X^{-3}\ll X^{-1}+X^{-3\xi}\log^3X\ll X^{-1}\log^3 X$$

for $\kappa_j\le X^{1-\xi}\log X$ and $\xi\ge 1/3$ (which is our assumption for this reason), it follows that only the values of κ_j for which $|\kappa_j-t|\ll_\varepsilon t^\varepsilon$ will make a non-negligible contribution. To complete the proof we need now (see the author's paper [8]) the bound contained in

LEMMA 3. *We have*

$$\sum_{K-G\le\kappa_j\le K+G}\alpha_jH_j^3(\tfrac12)\ll_\varepsilon GK^{1+\varepsilon}\quad(K^\varepsilon\le G\le K). \tag{4.12}$$

Thus we are left with the contribution which is, by (4.12),

$$\ll_\varepsilon t^{-1/2}\sum_{|t-\kappa_j|<t^\varepsilon}\alpha_jH_j^3(\tfrac12)X^{1/2-\sigma}\ll_\varepsilon t^{1/2+\varepsilon}X^{1/2-\sigma}.$$

Since $\sigma>\frac12$, the last expression is $\ll_\varepsilon t^{1-\sigma+\varepsilon}$ in the relevant range $X\gg t^{1-\varepsilon}$, and (2.12) follows. Our result is certainly not optimal, since by using (4.12) we have ignored the exponential factor in $L_\xi(x,\kappa_j)$ in (4.11) and the factor $x^{-i\kappa_j}$ in (4.10). On the other hand, there do not exist yet non-trivial estimates for exponential sums with $\alpha_jH_j^3(\frac12)$, which vitiates our efforts to improve on (2.12).

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